IMPLEMENTING STOCHASTIC PRODUCTS SELLING IN MOBILE GAMES: IS GACHA JUST GAMBLING?

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ABSTRACT

This paper studies the economic impacts of implementing the "Gacha" strategy in free-to-pay mobile games. The Gacha strategy allows game providers to sell stochastic products in games, such that the players pay in-game virtual currencies to obtain a random product. Various Gacha games have achieved financial success in recent years. One such Gacha game is Genshin Impact, an action role-playing game. After being released in September 2020, Genshin Impact had a revenue of more than $3 billion in its first year of release, the highest for any video game. Prior studies on Gacha games mainly focus on debating whether the Gacha games can be associated with gambling. To the best of our knowledge, the economic impacts of the Gacha on free-to-pay mobile games have not been explored yet. In this paper, we build a theoretical model to study how free-to-play mobile game providers benefit from the Gacha strategy. We show that, for the mobile games that follow the Game-as-a-Service revenue model, the Gacha strategy could lead to higher revenue than the commonly used Freemium strategy. We characterize a sufficient condition for the Gacha strategy to be optimal. We further show that the Gacha strategy converts non-paid players to payers gradually, and thus benefits the game providers in the long run.

Keywords: Mobile games; Freemium; Gacha; Game as a service (GaaS); Theoretical model

1. Introduction

In recent years, the mobile game industry has been growing tremendously. Annual revenues of mobile games in the United States have increased from $9 billion in 2012 to $41.7 billion in 2022 (Clement, 2022b). Technological advances, digital innovation, and nationwide lockdowns amidst the coronavirus (COVID-19) pandemic contribute to such rapid growth. In 2022, there are 156 million mobile video game users in the United States, which is higher than ever.

A dominant pricing strategy that has been adopted by a majority of mobile game providers is commonly known as free-to-play. Chapple (2020) shows that about 87% of mobile games were free-to-play in the Apple App Store in 2019. Genshin Impact, an action role-playing game, is one of the most successful free-to-play mobile games. After being released in September 2020, it had a first-year launch revenue of more than $3 billion in its first year of release, the highest for any video game, and a revenue of more than $4 billion as of March 2022 (Chapple, 2021). The uniqueness in the revenue model of Genshin Impact is that the game is free-to-play, and is monetized through "Gacha" game mechanics.

The gacha mechanic is similar to loot boxes. In Gacha games, a player can spend in-game virtual currency to draw a random character or a random weapon from a pool. Each pool typically contains one key product (a character or a weapon) that is attractive to the players. The key product, such as a strong character or a powerful weapon, can improve the involvement levels of the players who get these key products. Players can either purchase virtual currencies with real money, or earn virtual currencies by finishing in-game contents, challenges, or achievements.

Traditionally, there are two commonly used monetization methods for mobile games: in-game advertising and in-game purchases (Meng et al., 2021). The former method allows the game provider to monetize players' playtime, and the latter one, usually known as the "freemium" model, generates revenue by selling the premium content to the players. The freemium model is close to the Gacha model since usually the game providers of the freemium games also adopt in-game virtual currency and allow players to obtain virtual currency through finishing in-game content or purchasing. The major difference between the freemium model and the Gacha model is how the products are priced and sold. In traditional freemium games, the game provider sells deterministic products, i.e., the game provider sets prices (in virtual currency) for each product, and each player selects the product and pays the price. In the Gacha games, the game provider sells stochastic products, i.e., the game provider sets the price for a random draw, and each player pays the price to draw one product from a pool of a variety of products.

The Gacha game model began to be widely used in the early 2010s, particularly in Japan (Toto, 2020). It has been increasingly used in Chinese, Korean, and Western games (Nakamura, 2017). Genshin Impact is not the only Gacha game that achieves success. In the first quarter of 2022, the top five grossing Gacha games, Genshin Impact, Lineage W, Uma Musume Pretty Derby, Monster Strike, and Rise of Kingdoms, earn a revenue of $567 million, $272 million, $236 million, $197 million, and $185 million respectively (Clement, 2022a). The total quarterly revenue of these five games is more than $1.5 billion. Even though more and more mobile games achieve financial success, the prior literature has not provided enough results to explain how mobile game providers benefit from the Gacha model. The existing papers mainly focus on associating players purchasing behavior in Gacha games with problematic gambling. However, to the best of our knowledge, the economic impacts of the Gacha strategy have been unexplored.

In this paper, we develop a theoretic model to study how free-to-play mobile game providers can benefit from the Gacha strategy. In order to focus on the economic impacts of the Gacha strategy, we assume that all the players are rational in their decision making, to exclude any gambling-related factors, such as risk-seeking or microtransaction addiction, from our model. We compare the Gacha strategy to the traditional freemium strategy since both models are commonly adopted by free-to-play mobile game providers. The major difference between these two strategies is that the game provider sells deterministic products with the freemium strategy and sells stochastic products with the Gacha strategy.

Our major finding is that the Gacha strategy could be more profitable than the freemium strategy when the game provider adopts a game as a service (GaaS) revenue model, in which the game provider earns a continuing revenue crossing multiple time periods. We further compare the periodic revenue of the Gacha model with that of the freemium model. We find that, initially, the Gacha strategy may hurt the game provider's profitability since it allows some players to obtain some key products for free. Nevertheless, some other non-paid players may also get the key products for free, which may significantly increase their involvement levels in the game. As a result, a portion of the non-paid players will be converted to payers in late periods. Therefore, at the cost of losing some initial profits, the Gacha strategy helps to gradually convert non-paid players to payers, and thus benefits the game provider in the long run. As a comparison, we show that when the freemium strategy is adopted, even though the game provider earns higher revenues in early periods, it gradually becomes less and less profitable than the Gacha case. This is because, with the freemium strategy, all the payers always remain as payers and non-payers always remain as non-payer. The low conversion rate with the freemium strategy fits with the common observation in free-to-play games such that only 2%-3% of players are spending money in mobile games (ironSource, 2021).

Our study extends the prior literature on freemium models. We extend the current understanding of the freemium model that only allows deterministic product selling to the emerging Gacha model which allows stochastic product selling. We show that the Gacha strategy can efficiently convert non-paid players to payers, and thus benefit the game providers in the long run. We characterize a sufficient condition for the Gacha strategy to be optimal. It requires the game provider to follow the Game-as-a-service model to operate the game for multiple periods, and focus on the revenues in the long run without discussing the future revenues too much. Moreover, it also requires the game provider to carefully design and develop the key products so that the involvement levels of the players will be increased after obtaining these products. When these conditions are satisfied, the Gacha strategy could be an optimal choice for the game provider. Moreover, our proposed model allows us to incorporate GaaS, the popular revenue model for mobile games, into both the freemium and Gacha models. Our results help free-to-play mobile game providers to better understand the benefits of the Gacha model. We also provide useful guidelines for the game providers to determine when and how to adopt the Gacha strategy.

The rest of the paper is organized as follows. In the next section, we review the relevant prior works. In Section 3, we present our model settings and derive the optimal solution of the Benchmark case. In Section 4, we study the solutions and insights of the Gacha case. We discuss managerial insights and check for robustness in Section 5. Finally, we provide concluding remarks and possible future extensions in Section 6.

2. Literature Review

Prior literature has focused on player engagement in digital games being associated with factors like game content, environment, and personal preferences (Boyle et al., 2012). Player engagement for extended periods of time with ads congruent to the gaming content, increases advertisement platform's efficiency (De Pelsmacker et al., 2019). Most digital gaming content is offered for purchase to enhance the gaming experience (Soroush et al., 2018), while the purchase behavior is driven by enjoyment, monetary, and social identity values perceived by the player Wang et al. (2022). Game manufacturers attempting to appeal and capture broader audience, have given rise to free-to-play games that are structured to promote spending in addition to skilled gameplay (Howard, 2019). Digital free-to-play games increasingly include microtransactions with monetary features to hook the player and enhance engagement (Hamari et al., 2020, Howard, 2019). The Freemium pricing model used within these free-to-play games for premium content helps monetize playtime (Meng et al., 2021). Based on their time-sensitivity and gaming skills, players may resort to direct purchase of the Freemium content to enhance their experience because of multiple factors including impatience, unobstructed play, social interaction, and economical rationale (Evans, 2016; Hamari et al., 2020).

Popular games like Genshin Impact employ Gacha strategy that involves microtransactions with chance-determined or stochastic features and products in addition to regular deterministic ones (Nakamura, 2017). There have been concerns regarding gambling, predatory pricing, overspending, and regulatory challenge associated with chance-determined products and features offered in these games (King & Delfabbro, 2019; 2018, Petrovskaya & Zendle, 2021). However, there is inconclusive evidence of association of free-to-play games employing Gacha strategy with desire to gamble. While some researchers have associated purchase behavior in Gacha strategy with problematic gambling, especially in context of loot-boxes (Drummond & Sauer, 2018, Zendle & Cairns, 2018), others have not supported the concept (Griffiths, 2018; Macey & Hamari, 2018). For simplicity, we steer away from any gambling effects and focus on the effect of Gacha model as an extension of Freemium pricing strategy. Most often the premium game content can be purchased with in-game virtual currency using real currency or in-game player challenges (Guo et al., 2019). Virtual currencies in digital games are becoming popular and conversion rate with respect to real currency has been heavily studied in game mechanics and designs (Yamaguchi, 2004). Some researchers have also focused on virtual goods economy and virtual item sales (Lehdonvirta, 2009). Since there is no difference in how virtual currencies operate in Freemium and Gacha models, we choose not to delve in the intricacies of virtual world economies in this study.

Interaction with other players, sense of achievement, and community bonding are some other antecedents to player engagement in the digital games (Shi et al., 2015). Purchase intentions for virtual items in these digital games is influenced by the information exchange and social interactions online (Hsieh & Tseng, 2018). Skilled competition influences the experience and engagement of the player (Liu et al., 2013). When matched with similar skill level competitor, players earn achievement and reputation, but play longer when matched with same skill-level. Unobstructed play and competition significantly influence player engagement that further drives in-game purchase intentions and willingness to pay in free-to-play games (Pangaribuan et al., 2021). Researchers attribute player engagement behaviors to motivational affordances, features that offer players psychological stimulation and satisfaction, by facilitating interactions and competitions (Gupta et al., 2022; Yang et al., 2019; Tondello et al., 2019).

Contrary to expectations, all heavy users do not convert to paying users for these free-to-play games (Gupta et al., 2022). Those with superior skills compared to their competition in the game less likely become paying consumers while those struggling to advance or compete make in-app purchases. Some others merely collect extrinsic rewards to boost their experience and reputation in the game. While top 10 percent of paying users account for about 70 percent of the in-app purchase revenue, most of the players are not willing to pay driven by perceived fairness and aggressive monetizing tactics of game providers (Salehudin & Alpert, 2022). In this study, we explore conversion of non-paying players into paying players with use of Gacha strategy for in-app purchases.

Digital games have transitioned from a product design into customer-oriented experience-based services with a cultural shift towards player engagement and shared personal connections (Wilhelmsson et al., 2022). Game as a Service (GaaS) model offers several benefits to providers who can refresh content and add new content to original game long time after the initial release. This not only re-engages an experienced player back into the game, it provides sustained revenue from the same game with marginal investments. In this study, we explore how Gacha strategy interacts with GaaS model to influence player engagement, willingness-to-pay, and non-payer to payer conversion rates in free-to-play digital games.

3. Model Settings and the Benchmark Strategy

This model is developed from the game provider's perspective. Consider a monopoly game provider who offers a free-to-play game to a unit mass of players. We consider the following unique characteristics of Gacha mobile games:

Multiple periods and Game-as-a-service. Traditional digital games usually focus on the one-time revenue of selling the games. However, the free-to-play mobile games usually follow a continuing revenue model which is commonly known as game-as-a-service (GaaS) Model. In order to fit with the free-to-play mobile games that adopt GaaS revenue model, we build our model with N periods.

Free-to-Play Content and the key Products. With the GaaS model, a game provider releases new versions of a game following certain frequencies. Each new release usually contains a significant amount of new content. The new content typically contains free part and non-free part. For example, Genshin Impact releases a new version every one and half months. Each release contains new maps for the players to explore, new playing models, or special events, all of which are free for all the players. In addition, the release also includes some special products, such as special characters or weapons, that are not free to obtain. We refer to these special products as the key products. Some real-world examples of such special products in free-to-play mobile games include the five-star characters in Genshin Impact, and Legendary Crests in Diablo Immortal. In our model, we consider that the game provider develops a new key product with quality ξ and sells it in each release. Since our focus is to compare the pricing strategies, we consider ξ as an exogenous variable and treat its development costs as sunk costs.

The involvement level of the players. Players are heterogeneous in the levels of how they are getting involved in the game. Such a involvement level affects each player's willingness-to-pay (WTP) for the products that are sold in the game by the game provider. Let θ denote a player's involvement level to the game, where high θ represents high involvement level. At the beginning of the first period, θ of all the players is assumed to be uniformly distributed between 0 and 1.

The Impacts of the key Products. We consider two impacts on a player once he obtains a key product in a certain period. The short-term impact is that the player can directly get utility by owning this product. For example, in Genshin Impact, once a player owns a special character, he can play the game by using this character as his avatar to experience the new story line in this new release. He can also use the character's special skills to conquer hard challenges to access new content. We follow the standard product line design literature (see, for example, Mussa and Rosen, 1978; Jones and Mendelson, 2011) to model that, in each period, a player with involvement level θ gets utility $U(\theta, \xi) = \theta\xi$ by owning a key product with quality ξ. In the product line design literature, θ is commonly defined as the consumer type to heterogenized consumers' utilities for a product with a certain quality. In our model, we make θ be more meaningful in the context of mobile games by defining it as players' involvement level.

Mussa and Rosen (1978) and Jones and Mendelson (2011) model the product-line design problems only for one period. However, we model our problem with multiple periods. As a result, we need to consider how players' involvement levels change over time. One unique feature of the Gacha mobile games is that, owning a key product, such as a special character, may increase the involvement levels of the players in the long run. Gacha games providers typically invest a lot in designing and developing the key products. For a new special character, the game provider needs to invest in designing its 2D and/or 3D model, inviting popular voice actor for it, designing its new skill sets, developing its story lines, and determining its interactions with other characters. With such designs and developments, many players like the special characters they obtain and even treat them as idols, and thus get more involved in the games. Therefore, once a player with θ involvement level obtains a key product, his involvement level will be increased by $\beta(1 - \theta)$, *where* $0 \leq \beta \leq 1$ is defined as the attractiveness of the key product.

One piece of evidence that supports our argument is through observing players' activities outside the games. After obtaining some special characters in Gacha games, many players posted articles on online forums such as Reddit, or post videos on video sharing websites such as YouTube or Bilibili, to share their playing experience or feeling of owning the special characters. Interestingly, many of such player-generated articles or videos became very popular. For example, one player posted a video for the special character, Raiden Shogun in Genshin Impact, on YouTube (Row, 2022). This video gets 8.8 million views and more than 3,600 comments within a year. Similarly, there are a lot of player-generated videos for the special characters in Genshin Impact on Bilibili, a Chinese video shearing website, and many of which are among the top-viewed list with millions of views and thousands of comments. Many players even become fans of those special characters. All these observations show that owning the special characters in the Gacha game increases players' involvement levels in the long run. We need to clarify that such an impact in increasing players' involvement levels is a unique feature of the key products in the Gacha games. In contrast, the traditional free-to-play mobile games also sell products in games. For example, players could use virtual currency to purchase super bombs or other items in Candy Crush Saga to conquer some hard stages. However, we do not expect these types of products to significantly increase players' involvement levels. Therefore, even though the traditional free-to-play mobile game providers could adopt the Gacha strategy in selling their products, without redesigning the products for selling, they may not get the same benefits as the Gacha game providers do.

Players' Willingness-to-Pays (WTPs) for the key Products. As discussed above, once a player with involvement level θ obtains a key product in a certain period, there are two changes in his utility. First, he gets $\theta\xi$ as the direct utility in this period. Second, starting from next period, his involvement level will be increased by $\beta(1-\theta)$, and thus, he gets more utility from purchasing other key products in all the remaining periods. We call this long-term incremental utility as the indirect utility. Let $w(\theta)$ denote the WTP of a player with involvement level θ for the key product sold in each period. In our model, we assume that, in each period, a player's WTP for the key product is only determined by the direct utility, i.e., $w(\theta) = \theta\xi$. We exclude the indirect utility in players' WTPs because of the information asymmetry on the values of β and N. The game provider could estimate the attractiveness of the key product, β, by analyzing the data from all the players, but it should not be the same case for any player. Similarly, the game provider may know the total periods of the game, N, but will not release such information to the players. Without knowing β and N, players could not estimate the indirect utility from the key products in the long run. Later in Section 5, we will check the robustness of our results by relaxing this assumption to allow players to be forward-looking.

Pricing strategies for the key product. For Gacha games such as Genshin Impact, these key products usually make the major contribution towards the total revenue. Therefore, in this paper, we focus on the revenue that the game provider earns from the key products and consider pricing strategies to sell a key product in the game in each period. The game provider considers two pricing strategies to sell the key product: either to directly price the product and sell it, or to sell it as a stochastic product, i.e., a player pays for each draw and has a certain chance to get the key product within each draw. These two pricing strategies are defined as the "Benchmark" strategy, and the "Gacha" strategy respectively.

In-game virtual currency. It is a common practice for the game providers to price and sell the key products with in-game virtual currencies in mobile games (see, for example, Meng et al., 2021; Guo et al., 2019). Players can either purchase virtual currencies with real money or earn virtual currencies by finishing in-game contents, challenges, or achievements. Typically, the game provider uses the second method to provide free virtual currencies to all the players to motivate them, including the ones who only want to play the game for free, to continue to play the game. We remark that, it is also commonly observed that, the game provider may provide free virtual currencies to the players to motivate them to watch sponsored ads in order to earn additional revenues from the ads' sponsors. Since such a sponsored ads revenue model is not the focus of our paper, we assume that the game provider earns no additional revenue when providing free virtual currencies to the players.

In our model, we consider that the game provider follows the standard strategies to adopt virtual currencies. Moreover, we consider that the game provider gives each player a certain amount of virtual currencies for free in each period, and lets the players purchase additional virtual currencies with real money. In order to keep tractability, we include such adoptions with following simplification. In the Benchmark case, where the game provider chooses to sell the key product directly, A player can use the free virtual currencies he has on-hand and buy the remaining virtual currencies with real money so that he will have virtual currencies to buy the key product. Such a strategy is equivalent to providing no free virtual currencies to the players and pricing the key product with a reduced price in our settings[1]. Therefore, in this case, we simply let the game provide determine the final price a player needs to pay out of pocket to get the key product. On the other hand, where the game provider chooses to sell the key product with the Gacha strategy, each player first spends the virtual currencies he receives for free to draw the key product. Then only the players who fail to obtain the key product with the free draws determine whether to continue to draw with paying real money for each additional draw. In this case, it is equivalent to assuming that each player has a certain chance to obtain the key product for free, and then those who do not get the key product determine whether to purchase it at an expected price. With such settings, we avoid directly including virtual currencies in our model to unnecessarily increase the complexity of our analysis. Meanwhile, our model still fits with the common practice where the game providers adopt virtual currencies and provide free virtual currencies to the players. We summarize all the notations in Table 1.

[1] Here we assume that the price of the key product is higher than the total virtual currencies a player collect from all the multiple periods.

Table 1: List of Notations

Parameters				
N				Number of total periods
n				Index of each period, $n \in \{1, \ldots, N\}$
θ				Player's involvement level
ξ				Quality of the key product
β				Attractiveness of the key product
$w(\theta)$				WTP for the key product of a player with involvement level θ
δ^{n-1}				Time discount factor
Variables				
B Model	S Model	L Model	G Model	
p_d	p_d^*	p_d^*	p_g	Price of the key product
-	q	q_l	q_g	Probability of a successful free draw
-	$\gamma(n)$	γ	$\gamma_g(n)$	Conversion rate of non-payers to payers in each n
$g_b(n)$	$g_s(n)$	$g_l(n)$	$g_g(n)$	Group size of payers in each n
$\Pi_b(n)$	$\Pi_s(n)$	$\Pi_l(n)$	$\Pi_g(n)$	Provider's profit in each n
Π_{bt}	Π_{st}	Π_{lt}	Π_{gt}	Provider's total profit

B Model: Benchmark model, discussed in Section 3.1
S Model: Gacha model with exogenous price, discussed in Section 4.1
L Model: A lower bound model for the S model, discussed in Section 4.2
G Model: Gacha model with endogenous price, discussed in Section4.4

3.1. The Benchmark Strategy

In this scenario, the game provider sells a new key product with quality ξ in each period n. The game provider needs to determine the price, p_d, of selling the key product directly to all the players. We assume that p_d remains the same in all periods. This assumption fits with the common practice of the Gacha game industry. Typically, the game providers use the same pricing strategy in each period. Figure 1 shows the decision making and involvement level changing for a player who has an involvement level θ at the beginning of the period. In each period, a player with involvement level θ purchases a key product if he gets positive surplus from the product, i.e., if $w(\theta) = \theta\xi \geq p_d$. Therefore, the marginal player θ_1 who is indifferent in purchasing can be obtained as $\theta_1 = p_d / \xi$. At the end of this period, a player's involvement level increases to $\theta + \beta(1 - \theta)$ if he purchases the key product or remains the same otherwise. As a result, each player's purchasing decision remains the same in each of the N periods, i.e., the ones who purchase the key product in the previous period continue to purchase, and the ones who do not purchase in the previous period continue to not purchase. This result fits with the common observations for the majority of the mobile games that adopt the free-to-play (with in-app-purchase options) strategy. Goodbay (2021) reports that in free-to-play mobile games, there are always two groups of players. The first group of players always plays the games for free and never pays, and the second group of players regularly makes in-game purchases. Turning a non-payer to a payer is always considered as a critical task for free-to-play mobile game providers.

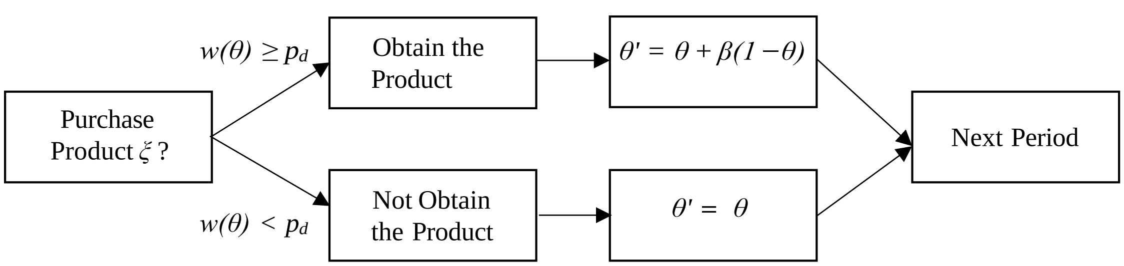


Figure 1: A Player's Decision Making and Involvement Level Changing in each period in the Benchmark Case

Now consider the total revenue for the game provider with the Benchmark strategy. Let the time discount factor be δ^{n-1}, i.e., one dollar in period n is worth δ^{n-1} dollars in the beginning of period 1. In each period, the game provider earns a profit of $(1-\frac{p_d}{\xi})p_d$, which leads to a total profit of $\sum_{n=1}^{N}\delta^{(n-1)}(1-\frac{p_d}{\xi})p_d$.Thus, we have the following result for this scenario:

Proposition 1 *When the game provider sells the key product directly, it is optimal to set the price of the key product as* $p_d^*=\frac{\xi}{2}$. *The optimal profit of each period n is* $\Pi_b^*(n)=\frac{\xi}{4}$ *and the total profit is* $\Pi_{bt}^*(n)=\frac{\xi(1-\delta^N)}{4(1-\delta)}$. *Moreover, the payer group size in each period n is* $g_b(n)=\frac{1}{2}$.

4. The Gacha Strategy

When the Gacha strategy is adopted, in each period, the seller sells a new key product as a stochastic product such that a player can use his in-game virtual currencies to draw the key product with a certain probability. The game provider provides a certain amount of free virtual currencies to each player, and the player can purchase more virtual currencies by using real money. As we discussed in subsection 3, we simplify such a scenario as the game provider determines a probability q that each player can get the key product for free, and a price p_g for selling the key product. Follow the common practice in Gacha games, we assume that the game provider adopts a stable strategy across periods and thus, both q and p_g remain the same in all the periods.

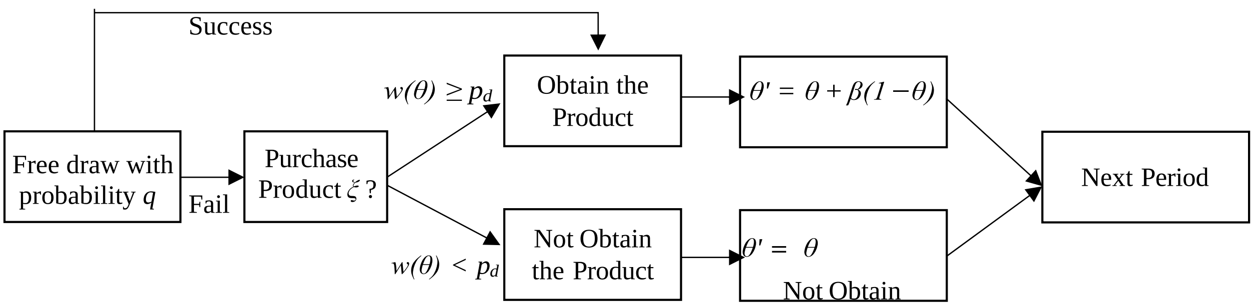


Figure 2: A Player's Decision Making and Involvement Level Changing in each period in the Gacha Case

Figure 2 shows a player's decision making and involvement level changing in a certain period with the Gacha strategy. In each period, a player with involvement level θ first draws the new key product for free with probability q. In case he fails the free draw, he purchases the product with price p_g if and only if $w(\theta)\geq p_g$. A player obtains the key product either through free draw or through purchasing. Regardless of whether the product is obtained through the free draw or through purchasing, the player's involvement level is increased by $\beta(1-\theta)$ at the end of this period. A player does not obtain the key product if he fails the free draw and chooses not to purchase it. In this case, his involvement level does not change at the end of this period.

4.1. The Gacha Strategy with Exogenous Price

We first consider the game provider adopts the same price as in the Benchmark case, i.e., to set $p_g=p_d^*=\frac{\xi}{2}$. Such a pricing strategy is suboptimal for the game provider. However, we will show that, even with such an exogenous price, the game provider can choose a certain probability to let each player obtain the key product for free, such that the game provider can get more profit than with the Benchmark strategy in the long run. Using such an exogenous pricing strategy allows us to reduce the complexity level of our analysis and to focus on exploring and analyzing the benefits of the Gacha strategy. We will study the true optimal solution of the Gacha case later in Section 4.4, in which the game provider endogenizes p_g as a decision variable.

At the beginning of the first period, each player uses the provided free virtual currencies to draw the key product and has a probability of q to successfully obtain it. Among the remaining $(1-q)$ players who have not obtained the key product through free draw[2], one with involvement level θ chooses to purchase it if he gets positive surplus from the product, i.e., if $w(\theta)=\theta\xi\geq p_d^*$. Regardless whether a player with involvement level θ obtains the key product through the free draw or through purchasing, his involvement level increases to $\theta+\beta(1-\theta)$. Therefore, the group size of the payers in the first period is $(1-q)(1-\frac{p_d^*}{\xi})$. Notice, since $q \geqslant 0$, this group size is smaller than the group size of the buyers in the Benchmark case.

[2] When there exists uncertainty, we maximize the expected profit for the game provider, and thus always use expectation calculation to simplify the analysis.

Now we consider the profit for the game provider. In each period n, a player with involvement level θ buys a key product if and only if he fails the free draw, and $w(\theta) \geq p_d^*$. Let $g_s(n)$ denote the group size of the payer in period *n*. $g_s(n)$ captures the total number of players who satisfy in period *n*. With $g_s(n)$, we can formulate the profit in period *n*, denoted as $\Pi_s(n)$, as:

$$\Pi_s(n) = g_s(n)(1-q)p_d^*, \qquad (1)$$

where $g_s(n)$ is the group size of the payers, *(1 − q)* captures the probability for each payer to fail the free draw, and p_d^* is the exogenous price of the key product. The total profit, denoted as Π_{st}, can be formulated as:

$$\Pi_{st} = \sum_{n=1}^{N} \delta^{n-1} \Pi_s(n) = \sum_{n=1}^{N} \delta^{n-1} g_s(n)(1-q)p_d^* \qquad (2)$$

Recall that, in the Benchmark case where there is no free draw, $g_s(n)$ takes constant value $\frac{1}{2}$ in each period *n*, because all the non-payers never purchase any key product and thus their WTPs never change. Now, with the Gacha strategy, each non-payer has a probability q to obtain the key product through the free draw. Once a non-payer successfully obtains the key product, he will be converted to a payer starting from the next period if his WTP is increased to be more than p_d^*. Therefore, in the Gacha case, $g_s(n)$ keeps increasing in *n*. In order to estimate $g_s(n)$, next, we analyze how some of the non-payers are converted to payers through the free draws. We start by analyzing the conversion rate in the first period. At the beginning of the first period, each player with an involvement level $\theta \epsilon \left[0, \frac{p_d^*}{\xi}\right]$ is a non-payer. Each non-payer has q probability to obtain the key product through the free draw, which increases his involvement level to $\theta + \beta(1-\theta)$. The marginal involvement level of the non-payer who will convert to a payer in the next period can be obtained by solving $\theta' + \beta(1-\theta') = \frac{p_d^*}{\xi}$. Thus, we have $\theta' = \frac{p_d^* - \beta\xi}{\xi(1-\beta)}$. When $\beta < \frac{p_d^*}{\xi}$, players with involvement levels in $\left[\theta', \frac{p_d^*}{\xi}\right]$ are converted from a non-payer to a payer. When $\beta \geq \frac{p_d^*}{\xi}$, each non-payer who obtains the key product through the free draw is converted to a payer. Let γ denote the conversion rate of non-payers to payers in the first period.

Then we have

$$\gamma = \begin{cases} \frac{\beta(\xi - p_d^*)}{p_d^*(1-\beta)} & \text{when } \beta < \frac{p_d^*}{\xi}, \\ 1 & \text{when } \beta \geq \frac{p_d^*}{\xi}. \end{cases} \qquad (3)$$

Equation (3) shows that, when the attractiveness of the key product, β, is lower than the threshold value $\frac{p_d^*}{\xi}$, $\frac{\beta(\xi - p_d^*)}{p_d^*(1-\beta)}$ of the non-payers will be converted to payers by getting the key product through the free draw. When β is greater than the threshold value $\frac{p_d^*}{\xi}$, then all the non-payers who succeed the free draw will be converted to payers, because, in this case, the lowest-type player (with $\theta = 0$) becomes a payer after getting the key product through the free draw.

Now we consider the conversion rate, denoted as γ_n, for $n \in \{2, \ldots, N\}$. Calculating γ_n for n $\geqslant$ 2 is not easy, since the players' involvement levels are not uniformly distributed anymore. For each γ_n, we need to split the non-payers into n groups. The first group is for the ones who fail the free draws in all the previous periods and thus, remain to be non-payers. Each of the remaining group, $n' \in \{2, \ldots, n\}$, is for the non-payers who succeed $n' - 1$ times free draws in the previous periods but his WTP is still below p_d^*. As a result, these non-payers remain to be non-payers at the beginning of period *n*. Let *m* be the minimum number of times of successful free draws needed for a non-payer with 0 involvement level to become a payer, i.e., $m = argmin_n \{1 - (1-\beta)^{n-1} \geq \frac{p_d^*}{\xi} | n \in \mathbb{N}\}$. Solving for *m*, we get $m = \left\lceil log_{1-\beta}(1 - \frac{p_d^*}{\xi}) + 1 \right\rceil$. Then, we have the following results for γ_n. The proofs of our main results are provided in the Appendix.

Proposition 2 *In each period n, the rate of converting non-payers to the payers through a successful free draw,* γ_n*, can be calculated as:*

$$\gamma_n = \begin{cases} \dfrac{\sum_{n'=1}^{n}\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'}}{\sum_{n'=1}^{n}\left(1-\frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}} & \text{for n} \in \{1,...,\text{m-1}\}, \\ \dfrac{\sum_{n'=1}^{m-1}\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'} + min\left\{\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{m}}\right), \left(1-\frac{\xi - p_d^*}{\xi(1-\beta)^{m-1}}\right)\right\}\binom{n-1}{m-1} q^{m}(1-q)^{n-m}}{\sum_{n'=1}^{m}\left(1-\frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}} & \text{for n} \in \{\text{m},...,\text{N}\}. \end{cases} \tag{4}$$

For each γ_n, we split the non-payers into *n* groups. The first group is for the ones who fail the free draws in all the previous periods and thus, remain to be non-payers. Each of the remaining group, $n' \in \{2, \dots, n\}$, is for the non-payers who succeed $n' - 1$ times free draws in the previous periods but his WTP is still below p^*. For n $\in$ {1,...,m-1}, the numerator captures the total number of non-payers that are converted to payers, because, for each n', the $\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'}$ term captures the number of non-payers who succeed $n' - 1$ times out of $n - 1$ total free draws. With the same explanation, the denominator captures the total number of non-payers. With γ_n, we obtain $g_s{}'(n)$, the number of payers as a function of *q*, for each *n*:

$$g'_s(n) = \begin{cases} \left(1 - \frac{p_d^*}{\xi}\right) & for \;\; n = 1, \\ 1 - (1 - g_s(n-1))(1 - \gamma_{n-1}) = 1 - \frac{p_d^*}{\xi}\prod_{n'=1}^{n-1}(1 - \gamma_{n'}) & for \;\; n \in \{2, \dots, N\}. \end{cases}$$

Finally, we can use γ_n and $g'_s(n)$ in Equation (2) to write the total profit for the game provider as a function of *q:*

$$\textstyle\prod'_{st} = \sum_{n=1}^{N} \delta^{n-1}(1-q)p_d^*\left(1 - \frac{p_d^*}{\xi}\prod_{n'=1}^{n-1}(1 - \gamma_{n'})\right)$$

Therefore, we have the following result regarding the optimal solution of this case. **Proposition 3** *When the game provider adopts the Gacha strategy and set the price of the key product same as that in the Benchmark case, the optimal probability of getting the key product through free draw is*

$$q^* = argmax_q \left\{\sum_{n=1}^{N} \delta^{n-1}(1-q)p_d^*\left(1 - \frac{p_d^*}{\xi}\prod_{n'=1}^{n-1}(1 - \gamma_{n'})\right)\right\} \qquad \forall\, 0 \le q \le 1 \tag{5}$$

where γ_n is given in Equation (4). Proposition 3 provides a solution approach to find the optimal probability for the free draw when the Gacha strategy is adopted. However, since each γ_n in formulation (5) is a complicated function of *q*, it is hard to derive structure insights from such a result. Next, we develop a solution approach that provides a lower-bound of the optimal solution of the problem. We will use this lower bound solution to derive a sufficient condition for the Gacha strategy to be more profitable than the Benchmark strategy. Moreover, it also allows us to discover managerial insights regarding the Gacha games.

4.2. A Lower-Bound for the Solution of the Gacha Case

One issue with the solution approach proposed in Proposition 3 is that the γ_n used in the formulation is a complicated function of *q*, making it hard to find an analytical optimal solution for the problem. In this subsection, we develop a lower-bound for the optimal solution of formulation (5). Such a lower-bound solution approach allows us to use first-order conditions to derive the optimal values for the free draw probability. As shown in Equation (4), γ_n is a complicated function of *q*. Next, we develop an underestimation of γ_n to make it independent of *q*. Later we will show that, such an underestimation leads to an approximation solution that significantly simplifies the problem. More importantly, the approximation solution always underestimates the optimal value of formulation (5), and thus provides a lower-bound for the optimal solution. We then use such a lower bound to discover a sufficient condition for the Gacha strategy to be more profitable than the Benchmark strategy. First, we present the following underestimation of γ_n.

Lemma 1 *In each period n, using γ that is defined in Equation (3) as the conversion rate underestimates the real conversion rate γ_n that is defined in Equation (4).*

With such an underestimated conversion rate, we could calculate the corresponding payer's group size $g_l(n)$, the game provider's profit in each period $\Pi_l(n)$ and the total profit Π_{lt} as follows.

$$g_l(n) = 1 - \frac{p_d^*}{\xi}(1 - q\gamma)^{n-1} \tag{6}$$

$$\textstyle\prod_l(n) = g_l(n)(1-q)p_d^* = 1 - \frac{p_d^*}{\xi}(1 - q\gamma)^{n-1}(1-q)p_d^* \tag{7}$$

$$\prod_{lt} = \sum_{n=1}^{N} \delta^{n-1} \prod_{l}(n) = \left(\frac{1-\delta^N}{1-\delta} - \frac{p_d^*\left(1-(\delta-\delta q\gamma)^N\right)}{\xi(1-\delta+\delta q\gamma)}\right)(1-q)p_d^* \quad (8)$$

Next, we solve for the optimal q_l^* that maximizes $\prod_{lt}$, and use q_l^* to find the optimal value $\prod_{lt}$, which can be used as a lower-bound for the true optimal of $\prod_{st}$.

Proposition 4 *When each γ_n in $\prod_{st}$ is replaced with an underestimated value γ in Equation (2), the corresponding optimal probability of getting the key product through the free draw is:*

$$q_l^* = argmax_q \left\{\left(\frac{1-\delta^N}{1-\delta} - \frac{p_d^*(1-(\delta-\delta q\gamma)^N)}{\xi(1-\delta+\delta q\gamma)}\right)(1-q)p_d^*\right\} \qquad \forall\, 0 \le q \le 1$$

$$Let\ \prod_{lt}^* = \left(\frac{1-\delta^N}{1-\delta} - \frac{p_d^*\left(1-(\delta-\delta q_l^*\gamma)^N\right)}{\xi(1-\delta+\delta q_l^*\gamma)}\right)(1-q_l^*)p_d^* \quad (9)$$

$\prod_{lt}^*$ *provides a lower bound for the optimal values of* $\prod_{st}$.

Proposition 4 shows that we can solve a bounded single-variable maximization problem to obtain the optimal value q_l^* for $\prod_{lt}$. Since the optimal value $\prod_{lt}^*$ provides a lower bound for the optimal value of $\prod_{st}$, it allows us to characterize a sufficient condition for the Gacha strategy to over-perform the Benchmark strategy. We formalize the sufficient condition for the Gacha strategy to be optimal as follows.

Proposition 5 It is optimal for the game provider to adopt the Gacha strategy when the following condition holds:

$$\left(\frac{1-\delta^N}{1-\delta} - \frac{p_d^*\left(1-(\delta-\delta q_l^*\gamma)^N\right)}{\xi(1-\delta+\delta q_l^*\gamma)}\right)(1-q_l^*) > \frac{1-\delta^N}{1-\delta}\left(1-\frac{p_d^*}{\xi}\right) \quad (10)$$

Proposition 5 provides a sufficient condition for the mobile game provider to check whether it is optimal to adopt the Gacha strategy in selling the key product. Next, we demonstrate our solution approach with examples where the Gacha strategy is optimal and discuss the insights of why Gacha could be optimal. Starting from now on, we will follow Proposition 4 to obtain q_l^* as the probability of drawing the key product for free and use q_l^* to obtain the solution for the problem. Such a simplified solution approach allows us to use first-order condition to derive the optimal value. As proved in Lemma 1, the obtained solution provides a lower bound for the true optimal value.

4.3. Examples and Managerial Insights

Example 1 *Consider a mobile game provider that operates a mobile game in N = 10 periods. Let δ = 0.9 and ξ = 1. In each period, the game provider sells a key product with* $\beta = 1/3$. When the mobile provider adopts the Benchmark strategy, we follow Proposition 1 to get the price of key product as $p_d^* = 0.5$. The game provider earns $\Pi_b^*(n) = 0.25$ in each period n and earns a total profit as $\Pi_{bt}^* \approx 1.628$. The payer group size in each period *n* is $g_b(n) = 0.5$.

On the other hand, when the game provider adopts the Gacha strategy with the same key product price $p_g = p_d^* = 0.5$, we first use Equation 3 to get $\gamma = \frac{\beta(1-p_d^*)}{p_d^*(1-\beta)}$. Then we follow Proposition 5 to get the probability of getting the key product through free draw as $q_l^* \approx 0.1223$. Give the q_l^* value, we follow Equations (6) and (7) to get the group size $g_l^*(n)$ and the profit $\Pi_l^*(n)$ for each period *n*. Moreover, we use Equation (8) to get the total profit for the game provider, $\Pi_{lt}^* \approx 1.706$. Compared to the total profit of the Benchmark case, the game provider improves its total profit by 4.75% by adopting the Gacha strategy.

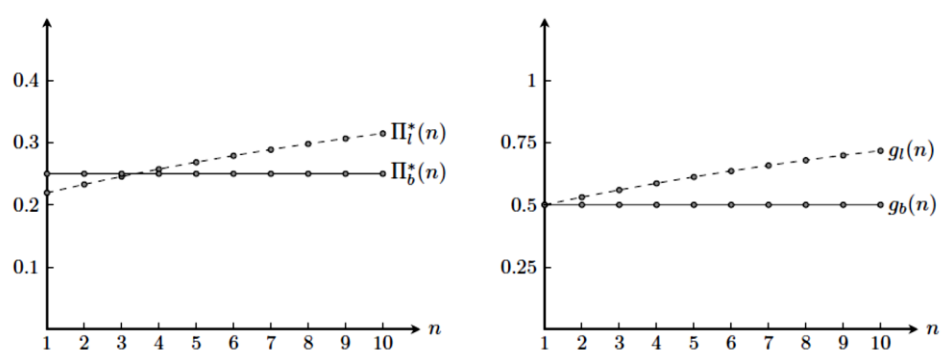


Figure 3: $\Pi_b^*(n)$ vs. $\Pi_l^*(n)$ and $g_b(n)$ vs. $g_l(n)$

In Figure 3, we show how the profit of each period n changes and how the group size of buyers changes for both strategies. When adopting the Benchmark strategy, $g_b(n)$, the group size of the payers, always remains the same in each period. As a result, $\Pi_b^*(n)$, the profit earned in each period also remains the same. When adopting the Gacha strategy, the game provider earns less profit in the first period than with the Benchmark strategy because of the following reason. At the beginning, the group sizes of the buyers are the same in both cases. With the Gacha strategy, q_l^* portion of the buyers get the key product through free draw, which reduces the profit of the game provider. However, the seller benefits from the free draw by converting a part of the non-buyers to buyers in each period, which continuously increases the game providers' profit in each period. In the fourth period, the game provider starts to earn more from the Gacha strategy, since the incremental profit earned from the converted payers exceeds the lost due to the free draw. Such a positive gap in profit keeps increasing in each of the remaining periods.

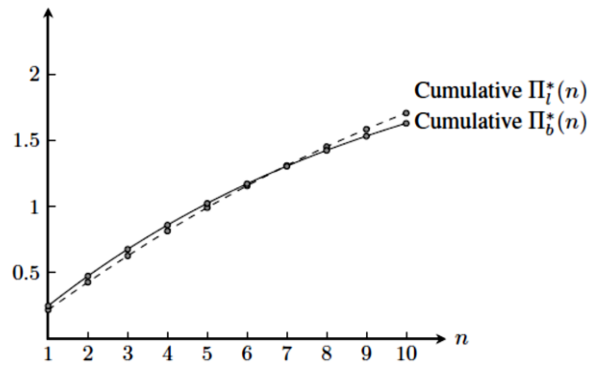


Figure 4: Cumulative $\Pi_b^*(n)$ vs. $\Pi_l^*(n)$

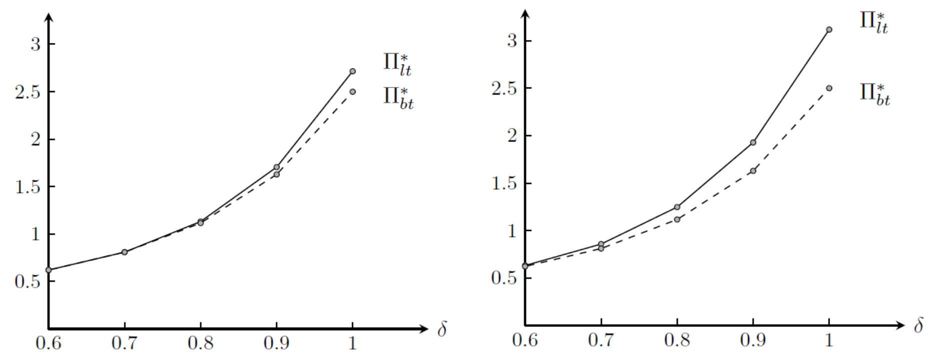
Figure 4 shows the cumulative profit in each period of both strategies. It shows that the game provider earns more with the Benchmark strategy in the beginning. However, the game provider starts to earn more with the Gacha strategy in the eighth period, and the difference in the profits of these two strategies keeps increasing in the remaining periods. Eventually, in the last period where N = 10, the cumulative profit of the Gacha strategy is 4.75% more than that of the Benchmark strategy. Moreover, we also observe that the group size of the payers has increased from 0.5 in the first period to about 0.72 in the tenth period, which means that about 44% of the non-payers are gradually converted to payers during these ten periods.

Table 2: The Impact of δ on Example 1

δ	1	0.9	0.8	0.7	0.6
q_l^*	0.1505	0.1228	0.0776	0.0057	0
Π_{lt}^*	2.7160	1.7056	1.1325	0.8098	—
Π_{bt}^*	2.5	1.6283	1.1158	0.8098	0.6212
$\Pi_{lt}^* - \Pi_{bt}^*$	0.2160	0.0773	0.0167	0.00005	—

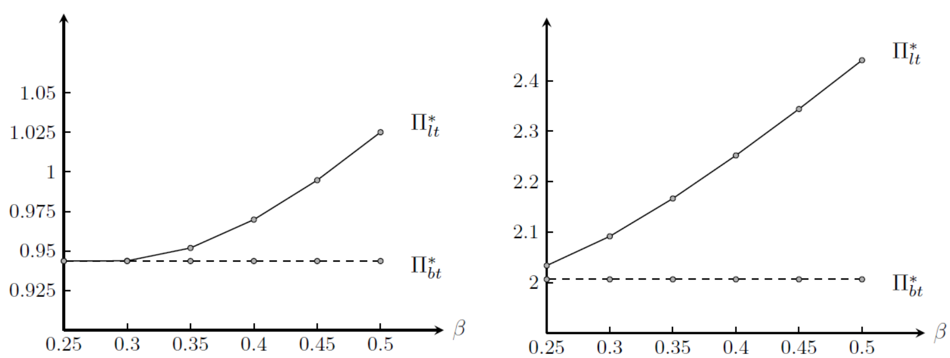
This example demonstrates how the game provider benefits from the Gacha strategy in the long term. In the short term, the Gacha strategy hurts the game provider since a portion of the buyers can get the key product through the free draw. As a payoff, the game provider converts a portion of the non-buyers to buyers. These newly converted payers bring extra profits in each of the remaining periods. Therefore, the Gacha strategy provides an efficient way for the game provider to convert non-payers to payers at cost of losing some profit from the current payers. When the game runs for long enough periods, the extra profit earned from the converted buyers can cover the lost due to the free draw. One interesting observation is that, although $\Pi_l^*(n)$ exceeds $\Pi_g^*(n)$ quickly and the gaps between these two keep increasing over *n*, as shown in Figure 3, the cumulative $\Pi_l^*(n)$ exceeds the cumulative $\Pi_g^*(n)$ at a much slower pace. This is due to the impact of the time discount factor δ^{n-1}. As discussed above, the Gacha strategy decreases the profits of the early periods and increases the profits of the late periods. However, δ^{n-1} discounts the future values and thus, weakens the benefits of the Gacha strategy. Table 2 shows the total profits of both strategies with different δ values. It clearly shows that small δ values are in favor of the Benchmark strategy over the Gacha strategy. When $\delta = 0.6$, the obtained $q_l^* = 0$, and thus, the game provider should adopt the Benchmark strategy.

We further consider the crossing impacts of β and δ on the total profits. In Figure 5 (a), we compare the impacts of δ on Π_{lt}^* and on Π_{bt}^* when $\beta = 1/3$. In Figure 5 (b), we show the same results when $\beta = 1/2$. It shows that the change in δ has greater impacts on Π_{lt}^* with higher β value. Similarly, in Figure 6 (a), we compare the impacts of β on Π_{lt}^* and on Π_{bt}^* when $\delta = 0.75$. In Figure 6 (b), we show the same results when $\delta = 0.95$. It shows that the change in β has greater impacts on Π_{lt}^* with higher δ value.



(a) $\beta = 1/3$ (b) $\beta = 1/2$

Figure 5: Impacts of δ on $\Pi_{bt}^*(n)$ and $\Pi_{lt}^*(n)$ with different β values



(a) $\delta = 0.75$ (b) $\delta = 0.95$

Figure 6: Impacts of β on $\Pi_{bt}^*(n)$ and $\Pi_{lt}^*(n)$ with different δ values

Another observation on Example 1 is that we obtained an optimal q_l^* that satisfies $0 < q_l^* \leq 1$ when the values of both N and β are reasonably large. However, when either β or N takes small value, it could happen that $q^* = 0$ becomes optimal, which means that the Benchmark strategy becomes optimal. In table 3, we let $\xi = 1$ and $\delta = 0.9$, and show the optimal solutions with different conversion rates γ and total periods N. We consider three conversion rates: a low conversion rate $\gamma = 1/3$ (when $\beta = 1/4$), a median conversion rate $\gamma = 1/2$ (when $\beta = 1/3$), and a high conversion rate $\gamma = 1$ (when $\beta = 1/2$). We also consider different total periods $N \in \{1, \ldots, 10\}$. For each scenario, the left table shows the optimal free draw probability q and the right table shows the optimal profit. In addition, the optimal profits of the Benchmark strategy with different N is shown in the second column of the right table. When it is optimal for the game provider to choose $q_l^* = 0$, it means that the game provider should not adopt the Gacha strategy and instead, should just adopt the Benchmark strategy.

Table 3: Optimal Free Draw Probabilities (q_l^*) and Profits with Different γ and N

N	q_l^*		
	$\gamma = 1/3$	$\gamma = 1/2$	$\gamma = 1$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0.088
5	0	0	0.131
6	0	0.028	0.150
7	0	0.068	0.158
8	0	0.094	0.162
9	0.023	0.111	0.164
10	0.052	0.123	0.164

N	Benchmark Profit	Gacha Profit		
		$\gamma = 1/3$	$\gamma = 1/2$	$\gamma = 1$
1	0.25	-	-	-
2	0.475	-	-	-
3	0.678	-	-	-
4	0.860	-	-	0.873
5	1.024	-	-	1.074
6	1.171	-	1.173	1.268
7	1.304	-	1.317	1.452
8	1.423	-	1.454	1.623
9	1.531	1.534	1.584	1.782
10	1.628	1.637	1.706	1.928

In this table, it shows that when $N \le 3$, the Benchmark strategy is always better than the Gacha strategy regardless of the conversion rate values. This result reveals the impact of the number of periods on the Gacha strategy. When the game provider operates a game with only a few periods, even though the Gacha strategy can help to convert non-payers to payers, the extra profit from the converted payers from later periods is not enough to cover the lost in profit caused by the free draws in the early periods. If the game provider can operate the game for more periods, the converted payers can generate more extra profit in later periods. Therefore, a game provider is more in favor of the Gacha strategy if the game is operated for more periods.

Now we discuss the observation on the impact of the conversion rate on the Gacha strategy. When the conversion rate is high ($\gamma = 1$), it means that each non-payer is converted to a payer once he succeeds the free draw in any period. In this case, it is optimal for the game provider to adopt the Gacha strategy when $N \ge 4$. The Gacha profit exceeds the Benchmark profit more with larger total periods. When $N = 10$, the Gacha profit is 18.37% higher than the Benchmark profit. When the conversion rate is median ($\gamma = 0.5$), it means that each non-payer has a 50% chance to be converted to a payer once he succeeds the free draw in any period. In this case, it is optimal for the game provider to adopt the Gacha strategy when $N \ge 6$. We also observe that the Gacha profit exceeds the Benchmark profit more with larger total periods. When $N = 10$, the Gacha profit is 4.75% higher than the Benchmark profit. When the conversion rate is low ($\gamma = 0.25$), it is optimal for the game provider to adopt the Gacha strategy when $N \ge 9$. When $N = 10$, the Gacha profit is only 0.55% higher than the Benchmark profit. Therefore, this result confirms our finding that a game provider is more in favor of the Gacha strategy if the game is operated for more periods. Moreover, it also shows that for any given N, the game provider is more in favor of the Gacha strategy with higher conversion rates. Later in Section 5, we will discuss the managerial insights in more detail.

4.4. The Gacha Strategy with Endogenous Price

In the previous section, we fix p_g, the price of the key product, to be the same as p_d^*, the optimal price of the Benchmark case, and only consider choosing the optimal probability q_g to let a player obtain the key product through the free draw. Clearly, using an exogenous price leads to a sub optimal profit for the game provider. In this section, we endogenize p_g as a decision variable into the Gacha model. The model remains almost the same as the one we treated in Section 4.1, except that the parameter p_d^* is replaced with the decision variable p_g. We remark that such a change furthermore increases the difficulty level of solving the problem, not only because the previous single-variable maximization problem now becomes one with two variables. More importantly, the conversion rate $\gamma_g(n)$ becomes a function that depends on both q_g and p_g. In this case, the conversion rate is:

$$\gamma_g(n) = \begin{cases} \dfrac{\sum_{n'=1}^{n}\left(\frac{\beta\xi - p_g}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q_g^{n'}(1-q_g)^{n-n'}}{\sum_{n'=1}^{n}\left(1-\frac{\xi - p_g}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q_g^{n'-1}(1-q_g)^{n-n'}} & \text{for n} \in \{1,\ldots,\text{m-1}\}, \\ \dfrac{\sum_{n'=1}^{m-1}\left(\frac{\beta\xi - p_g}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q_g^{n'}(1-q_g)^{n-n'} + min\left\{\left(\frac{\beta\xi - p_g}{\xi(1-\beta)^{m}}\right), \left(1-\frac{\xi - p_g}{\xi(1-\beta)^{m-1}}\right)\right\}\binom{n-1}{m-1} q_g^{m}(1-q_g)^{n-m}}{\sum_{n'=1}^{m}\left(1-\frac{\xi - p_g}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q_g^{n'-1}(1-q_g)^{n-n'}} & \text{for n} \in \{\text{m},\ldots,\text{N}\}. \end{cases} \tag{11}$$

The total profit for the game provider is:

$$\prod_{gt}(p_g, q_g) = \sum_{n=1}^{N} \delta^{n-1}(1-q_g)p_g\left(1-\frac{p_g}{\xi}\prod_{n'=1}^{n-1}(1-\gamma_g(n'))\right)$$

Then we have the following result for the optimal solution of this case.

Proposition 6 *When the game provider adopts the Gacha strategy and simultaneously determines the optimal price of the key product, p_g^*, and the optimal probability of getting the key product through free draw, q_g^*, we can obtain (p_g^*, q_g^*) as:*

$$(p_g^*, q_g^*) = \underset{p_g, q_g}{\operatorname{argmax}}\{\Pi gt(pg, qg) \quad s.t., 0 \le qg \le 1, 0 \le p\}. \tag{12}$$

The optimal profit of all the periods is $\Pi_{gt}^*(p_g^*, q_g^*)$.

Next, we use Proposition 6 to revisit Example 1. We keep all the settings in this example (ξ = 1, δ = 0.9, N = 10, and $\beta = 1/3$) except that we relax $p_g = p_d^* = 0.5$ and let both p_g and q_g be the decision variables. When following Proposition 6 to obtain the optimal solution, the first-order conditions of $\Pi_{gt}(p_g^*, q_g^*)$ are too complicated to yield closed-form solutions. Therefore, we use numerical solution approaches to obtain an approximation solution of $p_g^* \approx 0.525$ and $q_g^* \approx 0.1231$. The optimal total profit is $\Pi_{gt} \approx 1.710$. Recall that when we fix $p_g = p_d^* = 0.5$ and use an underestimation γ as the conversion rate in Section 4.3, we obtain $q_l^* \approx 0.1223$ and $\Pi_{st}^* = 1.706$. Therefore, the sub-optimal solution we obtained with $p_g = p_d^* = 0.5$ is close to the true optimal solution. The approach we proposed in Section 4.3 is easy to solve and yields an optimal value that is close to that of the true optimal solution.

5. Managerial Insights and Robustness Checks

Our results reveal important managerial insights regarding adopting the Gacha strategy in free-to-play mobile games. Following the examples and discussions provided in Section 4.3, we show that, although the Gacha strategy could be more profitable than the Benchmark strategy in selling the key products, it is not for every mobile game. We find three conditions for the Gacha strategy to be optimal. First, the mobile game needs to adopt the Game-as-a-Service model and operate the game for long enough periods. For those mobile games that focus on the revenues from only one period or only a few periods, it is more profitable to follow the Benchmark strategy to directly sell the products. Second, it also requires that the key products in the games can increase the involvement levels for the players who obtain those products. As we emphasize earlier in Section 3.1, such a requirement cannot be easily satisfied by many mobile games. It is a common practice for the Gacha game providers to invest heavily in designing and developing the key products (special characters) so that many players may like those special characters or even treat them as idols. In this way, players could get more involved in the game after obtaining such key products. It is different from the traditional free-to-play mobile games, such as Candy Crush Saga. For the products sold in the traditional mobile games, such as the super bombs or other items sold in Candy Crush Saga, we do not expect players' involvement levels to increase after obtaining these products. Thus, the Gacha strategy may not work for these traditional free-to-play games in selling their products without redesigning the products. Third, it also requires the game provider to not discount too much for future revenues. As we show earlier in our model, the usage of the free draws in the Gacha strategy is to convert non-payers to payers. As a result, it always decreases the revenues for the early periods and increases the revenues for the later periods. If a game provider discounts a lot for the future revenues, it diminishes the benefits of the Gacha strategy, and thus, could make it to be sub-optimal.

Gacha strategy is not a "magic" pricing strategy such that any game provider is guaranteed to increase their revenue by allowing their players to have some chances to obtain their key products for free. Instead, It requires the game provider to follow the Game-as-a-service model to operate the game for multiple periods and focus on the revenues in the long run without discussing the future revenues too much. Moreover, it also requires the game provider to carefully design and develop the key products so that the involvement levels of the players will be increased after obtaining these products. When these conditions are satisfied, the Gacha strategy could be an optimal choice for the game provider.

Next, we check the robustness of our results regarding a key assumption we made in the mode. Recall that we consider two impacts of obtaining the key products on the players: the short-term impact for the utility a player directly gets from the key product, and the long-term impact for the incremental involvement level that affects the utilities the player gets in later periods. Therefore, when obtaining a key product in a certain period *n*, a player gets $\theta\xi$ as direct utility in this period. In addition, since the player's involvement level will be increased by $\beta(1-\theta)$ at the end of this period, it leads to a total incremental utility of $\sum_{n'=n+1}^{N}\delta_p^{n'-n}\beta(1-\theta)\,\xi = \frac{1-\delta_p^{N-n+1}}{1-\delta_p}\beta(1-\theta)\xi$, which accounts for the total additional utility he might enjoy in all later periods. In this equation, δ_p^n is the time discount factor for the player, which can be different from the time discount factor for the game provider. In Sections 3 and 4, we assume that when a player makes a purchasing decision in each period, he only considers the utility he gets from the product in that period and ignores the incremental utilities he will get in later periods due to the incremental involvement level. We argue that, even though a player would like to be forward-looking in decision making, it is hard for him to precisely estimate the incremental utilities he will get in later periods, because it is hard for a player to get the perfect information that is needed for the estimation, such as the values of N and β. In order to check the robustness of our results on this critical assumption, we consider a more general WTP function, $w(\theta) = \theta\xi + \alpha(1 - \theta)\xi$, where α accounts for players' imperfect estimation of $\frac{1-\delta_p^{N-n+1}}{1-\delta_p}\beta$. Our original $w(\theta)$ function, $w(\theta) = \theta\xi$, is a special case when $\alpha = 0$, such that the players are purely myopic. In Section D in the Appendix, we derive our solution approaches for the Benchmark case and the Gacha case with such a general WTP function and check our main results. We find our main results remain the same.

6. Concluding Remarks

Many Gacha games have achieved financial success in recent years. Prior studies on Gacha games mainly focused on debating whether the Gacha games can be associated to gambling. This paper studies the economic impacts of implementing the Gacha strategy in free-to-pay mobile games. We built a theoretical model from the perspective of a free-to-play mobile game provider. We consider that the game provider follows the Game-as-a-Service revenue model to run a mobile game in multiple periods. Using such a model, we compare the Gacha strategy with the commonly adopted freemium strategy. We consider an exogenous price of the key product for the Gacha strategy and develop a lower-bound estimation for the optimal solution. Using the lower-bound estimation, we characterize a sufficient condition for the Gacha strategy to be optimal. It requires the game provider to follow the Game-as-a-service model to operate the game for multiple periods and focus on the revenues in the long run without discussing the future revenues too much. Moreover, it also requires the game provider to carefully design and develop the key products so that the involvement levels of the players will be increased after obtaining these products. When these conditions are satisfied, the Gacha strategy could be an optimal choice for the game provider. Finally, we discuss the optimal Gacha strategy when the price of the key product is endogenized. Our results provide insights regarding when and how mobile game providers could benefit from adopting the Gacha strategy.

One future extension area of this paper is to include a more sophisticated way of modeling players' Willingness-to-pays for the key products. Many other factors may also affect players' WTPs, such as the usage of the key products, word-of-mouth from other players, or competition among the players. Including these factors into the model can lead to richer results in understanding players' behaviors under the freemium or Gacha models. Another possible area of extension is to distinguish the impacts of Gacha on different categories of free-to-play mobile games, such as casual games, hard-core games, single-player games, multiple-players games, etc.

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APPENDIXES

Appendix A Proof of Proposition 2

Proof. For each γ_n, we split the non-payers into *n* groups. The first group is for the ones who fail the free draws in all the previous periods and thus, remain to be non-payers. Each of the remaining group, $n' \in \{2, \ldots, n\}$, is for the non-payers who succeed $n' - 1$ times free draws in the previous periods but his WTP is still below p^*. Next, we calculate *m*, the minimum number of times needed for a non-payer with 0 involvement to become a payer, *i.e.*, $m = argmin_n \left\{1 - (1-\beta)^{n-1} \geq \frac{p_d^*}{\xi} \mid n \in \mathbb{N}\right\}$. Solving for *m*, we get $m = \left\lceil log_{1-\beta}\left(1 - \frac{p_d^*}{\xi}\right) + 1\right\rceil$.

Now we calculate γ_n for each *n*. We start by calculating γ_n for $n \in \{1, \ldots, m-1\}$. Before the free draw, for each $n' \in \{1, \ldots, n\}$, the number of non-payers in group n', denoted as $g_{n'}$, can be calculated as $g_{n'} = (1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}})\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}$. The first term in $g_{n'}$, $(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}})$, captures the range of the original involvement levels for the players who remain to be non-payers with $n' - 1$ successful free draws. The value is obtained by solving for the marginal involvement level θ', that satisfies $1 - (1-\beta)^{n'-1}(1-\theta') = \frac{p_d^*}{\xi}$all the players with original involvement level in $[0, \theta']$ remain to be non-payers with $n' - 1$ successful free draws. The second term in $g_{n'}$, $\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}$, captures the probability that the non-payers have $n' - 1$ successful draws and $n - n'$ failed ones in the previous $n - 1$ draws. Next, we calculate the number of non-payers that will be converted to payers in each group n . We denote such a number as $c_{n'}$.In period *n*, *q* portion of the non-payers succeeds the free draw. Among this portion, the ones with original involvement levels in $[\theta'', \theta']$ will be converted to payers, where θ'' can be obtained by solving $1 - (1-\beta)^{n'}(1-\theta'') = \frac{p_d^*}{\xi}$. Solving, we get $\theta'' = (1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'}})$. Therefore, we have $c_{n'} = (\theta' - \theta'')\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'} = \left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'}$. The first term in $c_{n'}$captures the original involvement range of such converted payers. The second term in $c_{n'}$captures the probability that the converted payers succeed $n' - 1$ and fail $n - n'$ in the previous $n - 1$ draws, and in addition, succeed in period *n*. Given such $g_{n'}$ and $c_{n'}$, for $n \in \{1, \ldots, m-1\}$, $\gamma_n$can be calculated as

$$\gamma_n = \frac{\sum_{n'=1}^{n} c_{n'}}{\sum_{n'=1}^{n} g_{n'}} = \frac{\sum_{n'=1}^{n}\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'}}{\sum_{n'=1}^{n}\left(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}} \tag{13}$$

Next, we analyze the case for $n \in \{m, \ldots, N\}$. In this case, each group of $n' \in \{1, \ldots, m-1\}$ follows the same results as given in the previous paragraph. For group *m*, two situations may happen: either players in $[\theta'', \theta']$ are converted, or the whole group is converted. Thus, $c_m = \min\left\{\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^m}\right), \left(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{m-1}}\right)\right\}$. There will be no non-payers in groups *m + 1, . . .*, *N*, since any non-payer has already been converted to a payer once he succeeds the free draws for more than m times. Therefore, for $n \in \{m, \ldots, N\}$, $\gamma_n$can be calculated as

$$\gamma_n = \frac{\sum_{n'=1}^{m} c_{n'}}{\sum_{n'=1}^{m} g_{n'}}$$

$$= \frac{\sum_{n'=1}^{m}\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)\binom{n-1}{n'-1} q^{n'}(1-q)^{n-n'} + min\left\{\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^m}\right), \left(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{m-1}}\right)\right\}\binom{n-1}{m-1} q^m(1-q)^{n-m}}{\sum_{n'=1}^{m}\left(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}}\right)\binom{n-1}{n'-1} q^{n'-1}(1-q)^{n-n'}}$$

Appendix B Proof of Lemma 1

Proof. We first prove that for each $n \in \{1,\ldots,m-1\}$, it satisfies that $\gamma_n \geq \gamma$. Please refer to Section A for the definition of m, n', $c_{n'}$, and $g_{n'}$. For each $n' \in \{1,\ldots,n\}$, we have

$$\frac{c_{n'}}{g_{n'}} = \frac{\left(\frac{\beta\xi - p_d^*}{\xi(1-\beta)^{n'}}\right)}{\left(1 - \frac{\xi - p_d^*}{\xi(1-\beta)^{n'-1}}\right)} = \frac{\beta(\xi - p_d^*)}{p_d^*(1-\beta) - ((1-\beta)^n - (1-\beta))\xi} \geq \frac{\beta(\xi - p_d^*)}{p_d^*(1-\beta)} = \gamma$$

Therefore, we have

$\gamma_n = \frac{\sum_{n'=1}^{n} c_{n'}}{\sum_{n'=1}^{n} g_{n'}} \geq \frac{\sum_{n'=1}^{n} \gamma g_{n'}}{\sum_{n'=1}^{n} g_{n'}} = \gamma$

For $n \in \{m,\ldots,N\}$, we have $\frac{c_m}{g_m} = min\left\{\left(\frac{\beta(\xi - p_d^*)}{p_d^*(1-\beta)-((1-\beta)^m-(1-\beta))\xi}\right), 1\right\} \geq \frac{\beta(\xi-p_d^*)}{p_d^*(1-\beta)} = \gamma$. Therefore, we have

$$\gamma_n = \frac{\sum_{n'=1}^{m} c_{n'}}{\sum_{n'=1}^{m} g_{n'}} \geq \frac{\sum_{n'=1}^{m} \gamma g_{n'}}{\sum_{n'=1}^{m} g_{n'}} = \gamma$$

Appendix C Proof of Proposition 4

Proof. We first prove that, for each n, we have $g_s(n) \geq g_l(n)$. This holds by

$$g_s(n) = 1 - \frac{p_d^*}{\xi}\prod_{n'=1}^{n-1}(1-\gamma_{n'}) \geq 1 - \frac{p_d^*}{\xi}\prod_{n'=1}^{n-1}(1-\gamma) = g_l(n)$$

The first and last equality follows the definition of $g_s(n)$ and that of $g_l(n)$ respectively. The inequality holds by Lemma 1. Now consider using q_l^* as the value of q in Equation (2) to obtain a feasible objective value of the problem, denoted as $\prod_{st}'$ Then, consider the following chain of inequalities:

$$\prod_{st}^* \geq \prod_{st}' = \sum_{n=1}^{N} \delta^{n-1} g_s(n)(1-q_l^*)p_d^* \geq \sum_{n=1}^{N} \delta^{n-1} g_l(n)(1-q_l^*)p_d^* = \prod_{lt}^*$$

The first and last equality follows the definition of $\prod_{st}'$ and that of $\prod_{lt}^*$. The first inequality holds because $\prod_{st}^*$ is the optimal value of $\prod_{st}$ while $\prod_{st}'$ is one of its feasible values. The second inequality holds by $g_s(n) \geq g_l(n)$, as proved.

B Robustness Check: When $\boldsymbol{w(\theta) = (\theta + \alpha(1-\theta))\xi}$

In this section, we derive our solution approaches for the Benchmark case and the Gacha case and check our main results with a more general WTP function for the player: $\boldsymbol{w(\theta) = (\theta + \alpha(1-\theta))\xi}$. The WTP function we used in the main paper, $\boldsymbol{w(\theta) = \theta\xi}$, is a special case of this function with $\boldsymbol{\alpha = 0}$. For the Benchmark case, with $\boldsymbol{w(\theta) = (\theta + \alpha(1-\theta))\xi}$, the marginal player becomes $\theta' = \frac{p_d - \alpha\xi}{(1-\alpha)\xi}$. Then the benchmark solution is:

$$\boldsymbol{p_d^* = \frac{\xi}{2}}, \quad \boldsymbol{g_b(n) = \frac{1}{2(1-\alpha)}}, \quad \prod_{\boldsymbol{b}}^* = \boldsymbol{\frac{\xi}{4(1-\alpha)}}, \quad \prod_{\boldsymbol{bt}}^* = \boldsymbol{\frac{\xi(1-\delta^N)}{4(1-\alpha)(1-\delta)}}$$

For the Gacha case, we have $\gamma = \begin{cases} \frac{\beta(\xi - p_d^*)}{(p_d^* - \alpha\xi)(1-\alpha-\beta)} & when\ \beta < \frac{(1-\alpha)p_d^*}{\xi}, \\ 1 & when\ \beta \geq \frac{(1-\alpha)p_d^*}{\xi} \end{cases}$

The lower-bound solution for the Gacha case is:

$$q_l^* = argmax_q\left\{\left(\frac{1-\delta^N}{1-\delta} - \frac{(p_d^* - \alpha\xi)(1-(\delta-\delta q\gamma)^N)}{(1-\alpha)\xi(1-\delta+\delta q\gamma)}\right)(1-q)p_d^* \quad s.t., 0 \leq q \leq 1\right\},$$

$$g_l(n) = 1 - \frac{(p_d^* - \alpha\xi)}{(1-\alpha)\xi}(1-q_l^*\gamma)^{n-1},$$

$$\prod_l(n) = \left(1 - \frac{(p_d^* - \alpha\xi)}{(1-\alpha)\xi}(1-q_l^*\gamma)^{n-1}\right)(1-q_l^*)p_d^*,$$

$$\prod_{lt} = \left(\frac{1-\delta^N}{1-\delta} - \frac{(p_d^* - \alpha\xi)(1-(\delta-\delta q\gamma)^N)}{(1-\alpha)\xi(1-\delta+\delta q\gamma)}\right)(1-q_l^*)p_d^*$$

Consequently, the condition for the Gacha strategy to over-perform the Benchmark strategy becomes:

$$\left(\frac{1-\delta^N}{1-\delta} - \frac{(p_d^* - \alpha\xi)(1-(\delta-\delta q\gamma)^N)}{(1-\alpha)\xi(1-\delta+\delta q\gamma)}\right)(1-q_l^*) \geq \frac{(1-\delta^N)}{2(1-\alpha)(1-\delta)}$$